Solid-State and Soft Matter Photonics
Spring 2020 **Problem Set No.3**
Professor Y.Tomita **Due: July 15, 2020**

1. (30 points) **Coupling of two resonator modes**
In our latest lecture we learned the coupled-mode equations for two lossless resonators given by

$$\begin{aligned}\frac{da_1}{dt} &= i\omega_1 a_1 + \kappa_{12} a_2 \\ \frac{da_2}{dt} &= i\omega_2 a_2 + \kappa_{21} a_1,\end{aligned}$$

where $\kappa_{12} = -\kappa_{21}^*$. Assuming that $a_1(t)$ and $a_2(t)$ have an $\exp(i\omega t)$ dependence, where ω is a natural angular frequency of a coupled-resonator system, we found general solutions for a_1 and a_2 as

$$\begin{aligned}a_1(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}(Ae^{i\Omega_0 t} + Be^{-i\Omega_0 t}) \\ a_2(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}(Ce^{i\Omega_0 t} + De^{-i\Omega_0 t}),\end{aligned}$$

where $A \sim D$ are constants to be determined under the initial conditions, $a_1(0)$ and $a_2(0)$, and Ω_0 is given by

$$\Omega_0 = \sqrt{\frac{(\omega_1-\omega_2)^2}{4} + |\kappa_{12}|^2}.$$

Derive the following expressions for $a_1(t)$ and $a_2(t)$ in terms of $a_1(0)$ and $a_2(0)$:

$$\begin{aligned}a_1(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}\left[a_1(0)\left(\cos\Omega_0 t - i\frac{\omega_2-\omega_1}{2\Omega_0}\sin\Omega_0 t\right) + \frac{\kappa_{12}}{\Omega_0}a_2(0)\sin\Omega_0 t\right] \\ a_2(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}\left[\frac{\kappa_{21}}{\Omega_0}a_1(0)\sin\Omega_0 t + a_2(0)\left(\cos\Omega_0 t - i\frac{\omega_1-\omega_2}{2\Omega_0}\sin\Omega_0 t\right)\right].\end{aligned}$$

*Hint: You may use the Cramer's rule.