Solid-State and Soft Matter Photonics
Spring, 2020 **Problem Set No.4**
Professor Y.Tomita **Due: July 29, 2020**

1. (30 points) **Coupling of modes in space**
Coupled-mode equations that describe the lossless interaction between two optical wave modes a_1 and a_2 propagating along the z direction is given by

$$\begin{aligned} \frac{da_1}{dz} &= -i\beta_1 a_1 + \kappa_{12} a_2 \\ \frac{da_2}{dz} &= -i\beta_1 a_2 + \kappa_{21} a_1, \end{aligned}$$

where β_1 and β_2 are propagation constants for a_1 and a_2, respectively, which take either positive or negative values depending on whether the interaction is either codirectional or counterdirectional. And κ_{12} and κ_{21} are coupling constants for a_1 and a_2, respectively.

   (a) (10 points) Find that the relationships between κ_{12} and κ_{21} for both codirectional and counterdirectional coupling.

   (b) (10 points) Calculate the full gap width of the two propagation constants at $\omega = 0$ for two coupled copropagating modes when the dispersion relations of β_1 and β_2 are given by $a\omega$ and $b\omega$ ($a \neq b$), respectively.

   (c) (10 points) Show that the two dispersion curves for two coupled counterpropagating modes are hyperbolic (show the two foci points) and circle (show the radius) with for pure real values of β and for pure imaginary values of β, respectively, when the dispersion relations of β_1 and β_2 are given by $-\omega$ and ω, respectively.

2. (20 points) **Scattering matrix**
Prove Eq. (22), that is,

$$|\det[S]| = 1.$$

Hint: You can use the property on the determinant for the product of two matrices.

3. (20 points) **Transfer matrix**
Prove Eq. (35), that is,

$$[M] = \begin{pmatrix} t' - rr't^{-1} & rt^{-1} \\ -r't^{-1} & t^{-1} \end{pmatrix}.$$