Solid-State and Soft Matter Photonics
Spring, 2020 **Problem Set No.5**
Professor Y.Tomita **Due: August 12, 2020**

1. (20 points) **Chebyshev identity**
   Show that

$$U_N = 2\cos q\ell \, U_{N-1} - U_{N-2},$$

   where U_N is the Chebyshev polynominal given by

$$U_N = \frac{\sin(N+1)q\ell}{\sin q\ell}.$$

2. (40 points) **Photonic band structure**
   Consider a one-dimensional periodic medium such that

$$\epsilon(z) = \epsilon(z + \Lambda), \tag{1}$$

   where $\epsilon(z)$ is a dielectric constant and Λ is its period along the z axis. In a one-dimensional periodic medium $\epsilon(z)$ can be expressed, in terms of Fourier series, by

$$\epsilon(z) = \sum_{\ell} \epsilon_\ell \exp\left[-i\ell(2\pi/\Lambda)z\right], \quad (\ell = 0, \pm 1, \pm 2, \cdots). \tag{2}$$

   In this case an optical field in this periodic medium may be expressed in a Bloch wave form, as given by

$$E_K(z) = \sum_{\ell} A(K - \ell g) \exp\left(i\ell g z\right), \tag{3}$$

   where $A(K - \ell g)$ is the complex amplitude at a Bloch wavenumber $K - \ell g$, and

$$g = 2\pi/\Lambda. \tag{4}$$

   As shown in the lecture, we can derive the following linear coupled-mode equations for $A(K)$ and $A(k - g)$ from Eq. (3):

$$\begin{pmatrix} K^2 - \omega^2\mu\epsilon_0 & -\omega^2\mu\epsilon_1 \\ -\omega^2\mu\epsilon_{-1} & (K-g)^2 - \omega^2\mu\epsilon_0 \end{pmatrix} \begin{pmatrix} A(K) \\ A(K-g) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

   provided that $\epsilon(z)$ is real (*i.e.*, $\epsilon_1 = \epsilon_{-1}^*$) and that

$$|K - g| = K \tag{6}$$

   and

$$K^2 = \omega^2\mu\epsilon_0 \tag{7}$$

   are satisfied. Note that ϵ_0 shown in Eqs.(5) and (7) denotes the 0th-order Fourier component of $\epsilon(z)$.

(a) (10 points) Show that the Bragg condition given by Eq.(6) is satisfied when

$$K = \frac{1}{2}g \ (= \frac{\pi}{\Lambda}).$$

(b) (10 points) Show that the dispersion relation for this periodic medium is given by

$$(K^2 - \omega^2\mu\epsilon_0)[(K-g)^2 - \omega^2\mu\epsilon_0] - \omega^4\mu^2|\epsilon_1|^2 = 0.$$

(c) (10 points) Show that two values for ω^2 at $K = g/2$ are given by

$$\omega_{\pm}^2 = \frac{K^2}{\mu(\epsilon_0 \pm |\epsilon_1|)}.$$

(d) (10 points) Bloch waves at ω in the photonic band gap (PBG) do not propagate, but are evanescent (*i.e.*, exponentially decaying in the forward direction) owing to Bragg reflection. Show that PBG defined as $\Delta\omega_{gap} = |\omega_+ - \omega_-|$ is approximately given by

$$\Delta\omega_{gap} \approx \omega_0 \frac{|\epsilon_1|}{\epsilon_0},$$

where ω_0 is the center angular frequency in PBG and $\epsilon_0 \gg |\epsilon_1|$ is assumed.