Solid-State and Soft Matter Photonics
Spring, 2020 **Solution Set No.1**
Professor Y.Tomita **Due: June 03, 2020**

1. (20 points) **Symmetry of the conductivity tensor**
   (a) (10 points) Construct a symmetric tensor A^s and an asymmetric one A^a by A such that
   $$A^s = \frac{A + A^T}{2} \text{ and } A^a = \frac{A - A^T}{2},$$
   where "T" denotes the transpose operation, *i.e.*, $(A^T)_{ij} = (A)_{ji}$ $(i \neq j)$. Therefore, one finds that $(A^s)_{ij} = (A^s)_{ji}$ $(i \neq j)$ and $(A^a)_{ij} = -(A^a)_{ji}$ $(i \neq j)$ with $(A^a)_{ii} = 0$, so that
   $$A = A^s + A^a.$$
   It shows that any second-rank tensor A can be always expressed by the sum of A^s and A^a.
   (b) (10 points) We know from the result of (a) that $\tilde{\sigma}$ can be written the sum of its symmetric part $\tilde{\sigma}^s$ and asymmetric one $\tilde{\sigma}^a$. Then,
   $$\begin{aligned} P &= \boldsymbol{E} \cdot \tilde{\sigma} : \boldsymbol{E} \\ &= (\boldsymbol{E})_i (\tilde{\sigma})_{ij} (\boldsymbol{E})_j \\ &= (\boldsymbol{E})_i [(\tilde{\sigma}^s)_{ij} + (\tilde{\sigma}^a)_{ij}] (\boldsymbol{E})_j \\ &= (\boldsymbol{E})_i (\tilde{\sigma}^s)_{ij} (\boldsymbol{E})_j, \end{aligned}$$
   because $(\boldsymbol{E})_i (\tilde{\sigma}^a)_{ij} (\boldsymbol{E})_j = 0$ as $(\tilde{\sigma}^s)_{ij} = (\tilde{\sigma}^s)_{ji}$ $(i \neq j)$ and $(\tilde{\sigma}^a)_{ij} = -(\tilde{\sigma}^a)_{ji}$ $(i \neq j)$ with $(\tilde{\sigma}^a)_{ii} = 0$. Once again, Einstein's convention is understood here. Therefore, we prove that only the symmetric part of $\tilde{\sigma}$, $\tilde{\sigma}^s$, contributes to the power dissipation.

2. (10 points) **Time-reversal symmetry of the electric polarization vector**
   $$\boldsymbol{P}(\boldsymbol{r}, -\omega) = \int_{-\infty}^{+\infty} dt \boldsymbol{P}(\boldsymbol{r}, t) e^{-i\omega t} = \int_{+\infty}^{-\infty} dt \boldsymbol{P}(\boldsymbol{r}, -t) e^{i\omega t}.$$
   Since the time-reversed process goes from $+\infty$ to $-\infty$, the above result indicates that $\boldsymbol{P}(\boldsymbol{r}, -\omega)$ corresponds to the time-reversal form of $\boldsymbol{P}(\boldsymbol{r}, t)$ with $\omega \to -\omega$.

3. (10 points) **Symmetry of the complex electric susceptibility tensor**
   The condition that $\Re[i\omega \boldsymbol{E}(\boldsymbol{r}, \omega) \cdot \epsilon_0 \tilde{\chi}_e^*(\boldsymbol{r}, \omega) \boldsymbol{E}^*(\boldsymbol{r}, \omega)] = 0$ can be translated into the condition that $\Im[\boldsymbol{E}(\boldsymbol{r}, \omega) \cdot \epsilon_0 \tilde{\chi}_e^*(\boldsymbol{r}, \omega) \boldsymbol{E}^*(\boldsymbol{r}, \omega)] = 0$ ($\Im$ means an imaginary part). Thus,
   $$\Im[\boldsymbol{E} \cdot \tilde{\chi}_e^* \boldsymbol{E}^*] = \Im[E_i (\tilde{\chi}_e)_{ij}^* E_j^*] = \Im[E_i^2 (\tilde{\chi}_e)_{ii}^* + E_i E_j^* (\tilde{\chi}_e)_{ij}^*] = 0,$$
   where a summation over a repeated index i is understood (Einstein's convention). Since $(\tilde{\chi}_e)_{ii}$ (a diagonal element) is real for lossless media, The above equation can be rewritten as
   $$\Im[E_i E_j^* (\tilde{\chi}_e)_{ij}^*] = \frac{1}{2}[E_i E_j^* (\tilde{\chi}_e)_{ij}^* - E_i^* E_j (\tilde{\chi}_e)_{ij}] = 0,$$
   where the relation that $\Im(A) = 1/2(A - A^*)$ for any complex variable A is used. Since i and j are interchangeable due to the summation over i and j, we obtain
   $$E_i E_j^* [(\tilde{\chi}_e)_{ij}^* - (\tilde{\chi}_e)_{ji}] = 0,$$
   implying that $(\tilde{\chi}_e)_{ij}^* = (\tilde{\chi}_e)_{ji}$, *i.e.*, $\tilde{\chi}_e(\boldsymbol{r}, \omega)$ is Hermitian.