Solid-State and Soft Matter Photonics
Spring 2020
Professor Y.Tomita **Solution Set No.2**

1. (20 points) **Dirac identity**
In the following we shall prove the Dirac identity:

$$\left.\frac{1}{r \mp i\epsilon}\right|_{\epsilon\to 0} = \mathcal{P}\frac{1}{r} \pm i\pi\delta(r),$$

where $\mathcal{P}$ is the Cauchy's principal value.

Consider a complex function $f(z)$ that is analytical in the upper half-plane and approaches zero as $|z| \to \infty$. We next consider the following closed integral along the path shown in Fig. 1:

$$\lim_{\epsilon\to 0}\oint_C \frac{f(z)}{z-i\epsilon}dz = \lim_{\substack{\epsilon\to 0\\ R\to\infty\\ s\to 0}}\left[\int_{\psi_1} + \int_{-R}^{-s} + \int_{\psi_2} + \int_{+s}^{+R}\right]\frac{f(z)}{z-i\epsilon}dz. \quad (1)$$

The left-hand side of Eq. (1) is, according to Fig.2, given by

$$\begin{aligned}\lim_{\epsilon\to 0}\oint_C \frac{f(z)}{z-i\epsilon}dz &= \lim_{\epsilon\to 0}\left[\int_{\psi_1} + \int_{-\infty}^{+\infty}\right]\frac{f(z)}{z-i\epsilon}dz\\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{f(r)}{r-i\epsilon}dr,\end{aligned}$$

where we used that

$$\lim_{\epsilon\to 0}\int_{\psi_1}\frac{f(z)}{z-i\epsilon}dz = \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_0^{\pi}\frac{f(Re^{i\psi}+i\epsilon)}{Re^{i\psi}}iRe^{i\psi}d\psi = 0.$$

In the right-hand side of Eq.(1) we find that

$$\lim_{\epsilon\to 0}\int_{\psi_1}\frac{f(z)}{z-i\epsilon}dz = \lim_{R\to\infty}\int_{\psi_1}\frac{f(Re^{i\psi_1}+i\epsilon)}{Re^{i\psi_1}}iRe^{i\psi_1}d\psi_1 = 0,$$

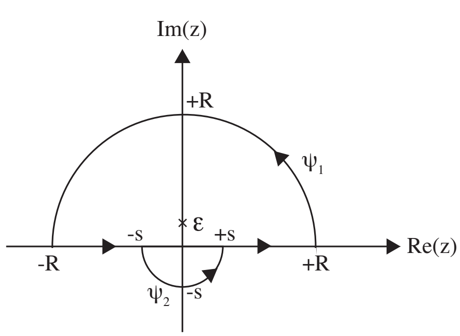

Fig. 1: Integration path.

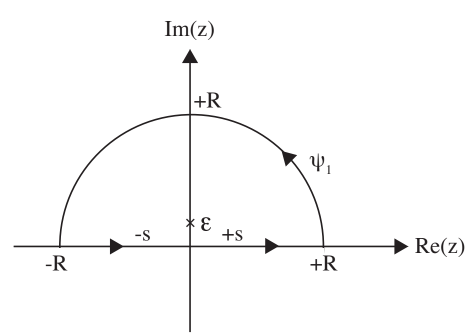

Fig. 2: Integration path.

and

$$\lim_{\epsilon\to 0}\int_{\psi_2}\frac{f(z)}{z-i\epsilon}dz = \lim_{\substack{\epsilon\to 0\\ s\to 0}}\int_{-\pi}^{0}\frac{f(se^{i\psi_2}+i\epsilon)}{se^{i\psi_2}}ise^{i\psi_2}d\psi_2 = i\pi f(0).$$

Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0}\oint_C\frac{f(z)}{z-i\epsilon}dz &= \lim_{\substack{R\to\infty\\ s\to 0}}\left[\int_{-R}^{-s}+\int_{+s}^{+R}\right]\frac{f(r)}{r}dr+i\pi f(0)\\ &= \mathcal{P}\int_{-\infty}^{+\infty}\frac{f(r)}{r}dr+i\pi f(0).\end{aligned}$$

Since

$$i\pi f(0) = i\pi\int_{-\infty}^{+\infty}\delta(r)f(r)dr,$$

we have

$$\left.\frac{f(r)}{r-i\epsilon}\right|_{\epsilon\to 0} = \mathcal{P}\frac{1}{r}+i\pi f(r)\delta(r)$$

for any $f(r)$. Thus, the following Dirac identity can be found:

$$\left.\frac{1}{r-i\epsilon}\right|_{\epsilon\to 0} = \mathcal{P}\frac{1}{r}+i\pi\delta(r).$$

Likewise, we have

$$\left.\frac{1}{r+i\epsilon}\right|_{\epsilon\to 0} = \mathcal{P}\frac{1}{r}-i\pi\delta(r).$$

Q.E.D.

2. (10 points) **Kramers-Kronig transformations**
In our previous lecture we derived that

$$\begin{aligned}\chi'(\omega) &= \mathcal{P}\int_{-\infty}^{\infty}\frac{d\nu}{\pi}\frac{\chi''(\nu)}{\nu-\omega}\\ \chi''(\omega) &= -\mathcal{P}\int_{-\infty}^{\infty}\frac{d\nu}{\pi}\frac{\chi'(\nu)}{\nu-\omega}.\end{aligned}$$

Multiplying a factor $\nu+\omega$ to both the denominator and the numerator of the integrant, we obtain that

$$\begin{aligned}\chi'(\omega) &= \mathcal{P}\int_{-\infty}^{\infty}\frac{d\nu}{\pi}\frac{\chi''(\nu)(\nu+\omega)}{\nu^2-\omega^2}\\ \chi''(\omega) &= -\mathcal{P}\int_{-\infty}^{\infty}\frac{d\nu}{\pi}\frac{\chi'(\nu)(\nu+\omega)}{\nu^2-\omega^2}.\end{aligned}$$

Since $\chi'(\nu)$ and $\chi''(\nu)$ are even and odd functions, respectively, we find that

$$\begin{aligned}\chi'(\omega) &= \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}d\nu\frac{\chi''(\nu)\nu}{\nu^2-\omega^2}\\ \chi''(\omega) &= -\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty}d\nu\frac{\chi'(\nu)}{\nu^2-\omega^2}.\end{aligned}$$

3. (20 points) **Quantum mechanical harmonic oscillators**

   (a) (10 points) We shall make use of the well-known result of quantum mechanical harmonic oscillators for $x_{nl} (\equiv < n|x|l >)$, which is given by

$$x_{nl} = \begin{cases} \sqrt{\frac{\hbar}{2m\omega}}(n+1)^{1/2} & \text{for } n = l+1, \\ \sqrt{\frac{\hbar}{2m\omega}} n^{1/2} & \text{for } n = l-1, \\ 0 & \text{otherwise,} \end{cases}$$

   where $\omega^2 = K/m$ (K is a spring constant).
   Considering the fact that $\epsilon_{nl} \equiv \epsilon_n - \epsilon_l = \omega(n+1/2) - \omega(l+1/2) = \omega(n-l)$, we obtain the oscillator strength $f_{nl} (\equiv 2m|x_{nl}|^2 \epsilon_{nl}/\hbar)$ as

$$f_{nl} = \begin{cases} n+1 & \text{for } n = l+1, \\ -n & \text{for } n = l-1, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) (10 points) We can calculate the total sum of f_{nl} by using the result in (a). It is straightword to show that

$$\sum_{n \neq l} f_{nl} = n + 1 - n = 1.$$