Solid-State and Soft Matter Photonics
Spring 2020
Professor Y.Tomita **Solution Set No.3**

1. (30 points) **Coupling of two resonator modes**
At $t = 0$ one finds

$$\begin{aligned} A + B &= a_1(0) \\ C + D &= a_2(0) \end{aligned}$$

Also, the coupled-wave equations at $t = 0$ provides

$$\begin{aligned} A\left(\frac{\omega_2 - \omega_1}{2} + \Omega_0\right) + B\left(\frac{\omega_2 - \omega_1}{2} - \Omega_0\right) + i\kappa_{12}C + i\kappa_{12}D &= 0 \\ C\left(\frac{\omega_1 - \omega_2}{2} + \Omega_0\right) + D\left(\frac{\omega_1 - \omega_2}{2} - \Omega_0\right) + i\kappa_{21}A + i\kappa_{21}B &= 0. \end{aligned}$$

We express them in a matrix form as given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{\omega_2-\omega_1}{2} + \Omega_0 & \frac{\omega_2-\omega_1}{2} - \Omega_0 & i\kappa_{12} & i\kappa_{12} \\ i\kappa_{21} & i\kappa_{21} & \frac{\omega_1-\omega_2}{2} + \Omega_0 & \frac{\omega_1-\omega_2}{2} - \Omega_0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} a_1(0) \\ a_2(0) \\ 0 \\ 0 \end{pmatrix}$$

We fist calculate the determinant of the 4×4 matrix in the above equation such that

$$\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{\omega_2-\omega_1}{2} + \Omega_0 & \frac{\omega_2-\omega_1}{2} - \Omega_0 & i\kappa_{12} & i\kappa_{12} \\ i\kappa_{21} & i\kappa_{21} & \frac{\omega_1-\omega_2}{2} + \Omega_0 & \frac{\omega_1-\omega_2}{2} - \Omega_0 \end{pmatrix} \equiv \det[\mathrm{M}] = -4\Omega_0^2.$$

Using Cramer's rule, we find that

$$\begin{aligned} A &= \frac{1}{\det[M]} \\ &= \begin{pmatrix} a_1(0) & 1 & 0 & 0 \\ a_2(0) & 0 & 1 & 1 \\ 0 & \frac{\omega_2-\omega_1}{2} - \Omega_0 & i\kappa_{12} & i\kappa_{12} \\ 0 & i\kappa_{21} & \frac{\omega_1-\omega_2}{2} + \Omega_0 & \frac{\omega_1-\omega_2}{2} - \Omega_0 \end{pmatrix} \\ &= -\frac{1}{2\Omega_0}\left[a_1(0)\frac{\omega_2 - \omega_1}{2} - a_1(0)\Omega_0 + ia_2(0)\kappa_{12}\right], \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{\det[M]} \\ &= \begin{pmatrix} 1 & a_1(0) & 0 & 0 \\ 0 & a_2(0) & 1 & 1 \\ \frac{\omega_2-\omega_1}{2} + \Omega_0 & 0 & i\kappa_{12} & i\kappa_{12} \\ i\kappa_{21} & 0 & \frac{\omega_1-\omega_2}{2} + \Omega_0 & \frac{\omega_1-\omega_2}{2} - \Omega_0 \end{pmatrix} \\ &= +\frac{1}{2\Omega_0}\left[a_1(0)\frac{\omega_2 - \omega_1}{2} + a_1(0)\Omega_0 + ia_2(0)\kappa_{12}\right], \end{aligned}$$

$$
\begin{aligned}
C &= \frac{1}{\det[M]} \\
&= \begin{pmatrix} 1 & 1 & a_1(0) & 0 \\ 0 & 0 & a_2(0) & 1 \\ \frac{\omega_2-\omega_1}{2}+\Omega_0 & \frac{\omega_2-\omega_1}{2}-\Omega_0 & 0 & i\kappa_{12} \\ i\kappa_{21} & i\kappa_{21} & 0 & \frac{\omega_1-\omega_2}{2}-\Omega_0 \end{pmatrix} \\
&= -\frac{1}{2\Omega_0}\left[a_2(0)\frac{\omega_1-\omega_2}{2} - a_2(0)\Omega_0 + ia_1(0)\kappa_{21}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
D &= \frac{1}{\det[M]} \\
&= \begin{pmatrix} 1 & 1 & 0 & a_1(0) \\ 0 & 0 & 1 & a_2(0) \\ \frac{\omega_2-\omega_1}{2}+\Omega_0 & \frac{\omega_2-\omega_1}{2}-\Omega_0 & i\kappa_{12} & 0 \\ i\kappa_{21} & i\kappa_{21} & \frac{\omega_1-\omega_2}{2}+\Omega_0 & 0 \end{pmatrix} \\
&= +\frac{1}{2\Omega_0}\left[a_2(0)\frac{\omega_1-\omega_2}{2} + a_2(0)\Omega_0 + ia_1(0)\kappa_{21}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_1(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}(Ae^{i\Omega_0 t} + Be^{-i\Omega_0 t}) \\
&= e^{i\frac{\omega_1+\omega_2}{2}t}[(A+B)\cos\Omega_0 t + i(A-B)\sin\Omega_0 t] \\
&= e^{i\frac{\omega_1+\omega_2}{2}t}\left[a_1(0)\left(\cos\Omega_0 t - i\frac{\omega_2-\omega_1}{2\Omega_0}\sin\Omega_0 t\right) + \frac{\kappa_{12}}{\Omega_0}a_2(0)\sin\Omega_0 t\right] \\
a_2(t) &= e^{i\frac{\omega_1+\omega_2}{2}t}(Ce^{i\Omega_0 t} + De^{-i\Omega_0 t}) \\
&= e^{i\frac{\omega_1+\omega_2}{2}t}[(C+D)\cos\Omega_0 t + i(C-D)\sin\Omega_0 t] \\
&= e^{i\frac{\omega_1+\omega_2}{2}t}\left[\frac{\kappa_{21}}{\Omega_0}a_1(0)\sin\Omega_0 t + a_2(0)\left(\cos\Omega_0 t - i\frac{\omega_1-\omega_2}{2\Omega_0}\sin\Omega_0 t\right)\right].
\end{aligned}
$$