Solid-State and Soft Matter Photonics
Spring, 2020
Professor Y.Tomita **Solution Set No.4**

1. (30 points) **Coupling of modes in space**
   (a) (10 points) Owing to the lossless interaction, the energy conservation of the two coupled modes a_1 and a_2 is hold such that
   $$\frac{dP}{dz} = 0, \tag{1}$$
   where P is the total power given by
   $$P = p_1|a_1|^2 + p_2|a_2|^2 \tag{2}$$
   in which $p_{1,2} = \pm 1$ depending on the interaction direction is either copropagating $(= +1)$ or counterpropagating $(= -1)$. Substitution of the coupled-mode equations into Eqs.(1) and (2) yields to
   $$p_1\kappa_{12} + p_2\kappa_{21}^* = 0. \tag{3}$$
   Therefore, one finds $\kappa_{12} = -\kappa_{21}^*$ and $\kappa_{12} = \kappa_{21}^*$ for codirectional and counterdirectional coupling, respectively.
   (b) (10 points) As shown in the class, the two propagation constants due to coupling are given by
   $$\beta_\pm = \frac{\beta_1 + \beta_2}{2} \pm \sqrt{\left(\frac{\beta_1 - \beta_2}{2}\right)^2 - \kappa_{12}\kappa_{21}}$$
   Using $\beta_1 = a\omega$ and $\beta_2 = b\omega$ with $\kappa_{12}\kappa_{21} = -|\kappa_{12}|^2$ for two coupled copropagating modes , one finds $\beta_\pm = \pm|\kappa_{12}|$ at $\omega = 0$. Therefore, the full gap width of the two propagation constants at $\omega = 0$ is given by
   $$|\beta_+ - \beta_-| = 2|\kappa_{12}|.$$
   (c) (10 points) Using $\beta_1 = -\omega$ and $\beta_2 = \omega$ with $\kappa_{12}\kappa_{21} = |\kappa_{12}|^2$ for two coupled counterpropagating modes , one finds $\beta_\pm = \pm\sqrt{\omega^2 - |\kappa_{12}|^2}$.
   For pure real values of β for $\beta_\pm$, *i.e.*. $|\omega| \geq |\kappa_{12}|$, one finds $\omega^2 - \beta^2 = |\kappa_{12}|^2$, showing hyperbolic curves with foci points of $(\omega, \beta) = (\pm\sqrt{2}|\kappa_{12}|, 0)$.
   For pure imaginary values of β for $\beta_\pm$, *i.e.*. $|\omega| \leq |\kappa_{12}|$, one finds $\beta_\pm = o\sqrt{|\kappa_{12}|^2 - \omega^2}$, indicating $[\mathrm{Im}(\beta)]^2 + \omega^2 = |\kappa_{12}|^2$. It shows a circle with the radius of $|\kappa_{12}|$.

2. (20 points) **Scattering matrix**
   Using the property that the determinant of the product of two matrices [A] and [B] equals the product of their determinants, one finds from Eq. (20) that
   $$\det[\mathrm{S}]^\dagger[\mathrm{S}] = \det[\mathrm{S}]^\dagger\det[\mathrm{S}] = 1.$$

On the other hand,

$$\mathrm{det[S]}^\dagger = \mathrm{S}_{11}^*\mathrm{S}_{22}^* - \mathrm{S}_{12}^*\mathrm{S}_{21}^* = (\mathrm{det[S]})^*.$$

Therefore, one finds that

$$\mathrm{det[S]}^\dagger\mathrm{[S]} = \mathrm{det[S]}^\dagger\mathrm{det[S]} = |\mathrm{det[S]}|^2 = 1,$$

implying that

$$|\mathrm{det[S]}| = 1.$$

You can also use Eq. (21) and $|\mathrm{det[S]}|^2 = |\mathrm{S}_{11}\mathrm{S}_{22} - \mathrm{S}_{12}\mathrm{S}_{21}|^2$ to prove that $|\mathrm{det[S]}| = 1$.

3. (20 points) **Transfer matrix**
Using the scattering matrix $[S]$, one can express Ψ_L^- and Ψ_R^+ in terms of Ψ_L^+ and Ψ_R^- such that

$$\begin{pmatrix}\Psi_L^- \\ \Psi_R^+\end{pmatrix} = \begin{pmatrix}S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix}\begin{pmatrix}\Psi_L^+ \\ \Psi_R^-\end{pmatrix} = \begin{pmatrix}r' & t \\ t' & r\end{pmatrix}\begin{pmatrix}\Psi_L^+ \\ \Psi_R^-\end{pmatrix},$$

from which one can find that

$$\Psi_L^- = r'\Psi_L^+ + t\Psi_R^- \quad \text{and} \quad \Psi_R^+ = t'\Psi_L^+ + r\Psi_R^-.$$

Rewriting the above equations for Ψ_R^+ and Ψ_R^- in terms of Ψ_L^+ and Ψ_L^-, one finds that

$$\begin{pmatrix}\Psi_R^+ \\ \Psi_R^-\end{pmatrix} = \begin{pmatrix}t' - rr't^{-1} & rt^{-1} \\ -r't^{-1} & t^{-1}\end{pmatrix}\begin{pmatrix}\Psi_L^+ \\ \Psi_L^-\end{pmatrix},$$

showing that

$$[M] = \begin{pmatrix}t' - rr't^{-1} & rt^{-1} \\ -r't^{-1} & t^{-1}\end{pmatrix}.$$