Solid-State and Soft Matter Photonics
Spring, 2020 **Solution Set No.5**
Professor Y.Tomita **Due: August 12, 2020**

1. (20 points) **Chebyshev identity**
Using the relation given by $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$, one can write
$$U_N = \frac{\sin(N+1)q\ell}{\sin q\ell} = \frac{\sin N\ell \cos q\ell + \cos Nq\ell \sin a\ell}{\sin q\ell}$$
and
$$U_{N-2} = \frac{\sin(N-1)q\ell}{\sin q\ell} = \frac{\sin N\ell \cos q\ell - \cos Nq\ell \sin a\ell}{\sin q\ell}.$$
Summing both sides of the above equations, one finds
$$U_N + N_{N-2} = \frac{2\cos q\ell \sin Nq\ell}{\sin q\ell} = 2\cos q\ell U_N,$$
by which one proves that
$$U_N = 2\cos q\ell\, U_{N-1} - U_{N-2}.$$

2. (40 points) **Photonic band structures**

(a) (10 points)
$$|K-g|^2 = K^2.$$
$$g(g-2K) = 0.$$
Thus, we have
$$K = \frac{1}{2}g \ (= \frac{\pi}{\Lambda}),$$
where we used $g = 2\pi/\Lambda$.

(b) (10 points) The condition that both $A(K)$ and $A(K-g)$ shown in Eq.(5) have non-zero values is
$$det \begin{pmatrix} K^2 - \omega^2\mu\epsilon_0 & -\omega^2\mu\epsilon_1 \\ -\omega^2\mu\epsilon_{-1} & (K-g)^2 - \omega^2\mu\epsilon_0 \end{pmatrix} = 0,$$
leading to
$$(K^2 - \omega^2\mu\epsilon_0)[(K-g)^2 - \omega^2\mu\epsilon_0] - \omega^4\mu^2|\epsilon_1|^2 = 0,$$
where we used $\epsilon_1 = \epsilon_{-1}^*$.

(c) (10 points) Plug $g = 2K$ in the dispersion relation to get
$$\omega_\pm^2 = \frac{K^2}{\mu(\epsilon_0 \pm |\epsilon_1|)}.$$

(d) (10 points) Put $\omega_\pm = \omega_0 \pm \Delta\omega$, where ω_0 is the center angular frequency in PBG and $\Delta\omega$ is the half width of $\Delta\omega_{gap}$.
$$\begin{aligned} \omega_+^2 - \omega_-^2 &= (\omega_+ - \omega_-)(\omega_+ + \omega_-) \\ &= (\omega_0 + \Delta\omega - \omega_0 + \Delta\omega)(\omega_0 + \Delta\omega + \omega_0 - \Delta\omega) \\ &= 2\Delta\omega \cdot 2\omega_0. \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta\omega_{gap} &= 2\Delta\omega \\
&= \frac{1}{2\omega_0}(\omega_+^2 - \omega_-^2) \\
&= \frac{1}{2\omega_0}(\epsilon_0\omega_0^2)\left(\frac{1}{\epsilon_0 - |\epsilon_1|} - \frac{1}{\epsilon_0 + |\epsilon_1|}\right) \\
&= \frac{\epsilon_0\omega_0}{2}\frac{2|\epsilon_1|}{\epsilon_0^2 - |\epsilon_1|^2} \\
&\approx \omega_0\frac{|\epsilon_1|}{\epsilon_0},
\end{aligned}$$

where we used $|\epsilon_1|/\epsilon_0 \ll 1$ so that $1/(\epsilon_0^2 - |\epsilon_1|^2) = 1/\epsilon_0^2(1 - |\epsilon_1|^2/\epsilon_0^2) \approx 1/\epsilon_0^2$. (If we keep the term $|\epsilon_1|^2/\epsilon_0^2$ in the approximation, then $1/(\epsilon_0^2 - |\epsilon_1|^2) \approx (1 + |\epsilon_1|^2/\epsilon_0^2)/\epsilon_0^2$. In this case $\Delta\omega_{gap} \approx \omega_0(1 + |\epsilon_1|^2/\epsilon_0^2)|\epsilon_1|/\epsilon_0$, which contains the correction of the order of $|\epsilon_1|^3/\epsilon_0^3$. Such a correction can be safely neglected in our case.)